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A Natural One-Form for the Schouten Concomitant

F. J. Bloore and M. Assimakopoulos

Department of Applied Mathematics and Theoretical Physics, The University, Liverpool, L69 3BX, England

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The Poisson bracket in classical mechanics arises from the existence of a natural one-form on a cotangent bundle. The Schouten concomitant of two symmetric contravariant tensor fields is closely related to the Poisson bracket. We show that it arises in an analogous way from a natural one-cochain, where the chains are chains of derivations from the module of symmetric contravariant tensor fields into itself.

1. INTRODUCTION

In classical mechanics, the phase space of a dynamical system is the cotangent bundle T^*M of its configuration manifold M. The observables are real C^∞ functions on T^*M, and form an associative commutative algebra $F(T^*M)$ under pointwise multiplication. By using the natural 1-form θ on T^*M (Abraham and Marsden, 1967), this algebra can also be furnished with a Lie product called the Poisson bracket,

$$\{f, g\} = d\theta(X_f, X_g), \qquad f, g \in F(T^*M)$$

where $X_f = (df)^\#$ is the Hamiltonian vector field generated by f.

The algebra $F(T^*M)$ has a graded subalgebra $F_0(T^*M)$ which consists of functions which are polynomial in momentum. To obtain this, we restrict the coordinate charts on T^*M to be of the form $(q^1, \ldots, q^n, p_1, \ldots, p_n)$, where the q^i are coordinates on M and the p_i are the components of the covector $p_i\, dq^i$ referred to the basis dq^i of the cotangent plane at q. The elements of $F_0(T^*M)$ are then polynomials in the p_i,

$$\sum S^{i_1 \cdots i_s}(q) p_{i_1} p_{i_2} \cdots p_{i_s}$$

with coefficients which are components of C^∞ symmetric contravariant

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tensor fields on M. The grade of an element is its degree in p. This filtration of $F_0(T^*M)$ is coordinate independent for the charts as restricted above.

In some formulations of quantization, only the elements of $F_0(T^*M)$ are assumed to correspond to quantum mechanical observables. It is then convenient to work not directly with $F_0(T^*M)$ but with the isomorphic algebra $\mathscr{A}$ of symmetric contravariant tensor fields on M. Here the pointwise associative commutative product, the Poisson bracket, and the grading of $F_0(T^*M)$ are replaced by, respectively, the symmetrized tensor product, the Schouten concomitant, and the tensor valence. We give details in Section 2. When working with $\mathscr{A}$ one need not mention T^*M but may concentrate on the geometrical properties of the configuration manifold M itself.

Now for $F(T^*M)$, it is an important mathematical fact that the existence of the Poisson bracket is due to the existence of the natural 1-form θ on T^*M. It may be helpful, therefore, in understanding the structure of the Schouten concomitant, to trace explicitly this 1-form through the isomorphism between $F_0(T^*M)$ and $\mathscr{A}$. That is the objective of the present note. We begin with the associative commutative algebra $\mathscr{A}_0$ of symmetric contravariant tensor fields on M furnished with the symmetrized tensor product. Unlike $F_0(T^*M)$, this algebra is not an algebra of functions on a manifold and so we cannot set up homology chains involving vector fields. Instead we must use derivations of $\mathscr{A}_0$, as described by Hermann (1973). We then show that $\mathscr{A}_0$ possesses a natural coderivation or 1-cochain $\tilde{\theta}$, which maps derivations on $\mathscr{A}_0$ into $\mathscr{A}_0$. This cochain has a nondegenerate exterior derivative $\tilde{d}\tilde{\theta}$, which we use to associate with each element of $\mathscr{A}_0$ the "Hamiltonian derivation" which it generates. We then define the Schouten concomitant using $\tilde{\theta}$ in exactly the same way as the Poisson bracket is defined using θ.

2. THE SCHOUTEN CONCOMITANT

Let $T^{(s)}M$ be the linear space of real fully symmetric contravariant C^∞ tensor fields S on M, with valence $v(S) = s$. Let $q^1, \ldots, q^n, p_1, \ldots, p_n$ be local coordinates of T^*M as described in the introduction. Denote by $C(S)$ the homogeneous function of degree $v(S)$ in the p's,

$$C(S) = S^{i_1 \ldots i_s}(q) p_{i_1} p_{i_2} \cdots p_{i_s}$$

The Schouten concomitant (Sommers, 1973) $[S, T]$ is an element of $T^{(r)}M$ where $r = v(S) + v(T) - 1$, related to the Poisson bracket by

$$\{C(S), C(T)\} = -C([S, T])$$

In terms of components,

$$[S, T]^{i_1 \cdots i_{s+t-1}} = sS^{r(i_1 \cdots i_{s-1}} \partial_r T^{i_s \cdots i_{s+t-1})} - tT^{r(i_1 \cdots i_{t-1}} \partial_r S^{i_t \cdots i_{t+s-1})} \quad (2.1)$$

where bracketed suffices are symmetrized and $\partial_r \equiv \partial/\partial q^r$. The direct sum $\mathscr{A} = \bigoplus_{m=0}^{\infty} T^{(m)}M$ is a Lie algebra with respect to the Schouten concomitant. The map $S \to C(S)$ gives a homomorphism $\mathscr{A} \to F(T^*M)$. Ordinary multiplication in $F(T^*M)$ is imaged in $\mathscr{A}$ by the symmetrized outer product $S \cap T \in T^{(s+t)}M$,

$$C(S)C(T) = C(S \cap T)$$
$$(S \cap T)^{i_1 \cdots i_{s+t}} = S^{(i_1 \cdots i_s}T^{i_{s+1} \cdots i_{s+t})}$$

Denote by $\mathscr{A}_0$ the linear space $\mathscr{A}$ furnished with $\cap$ multiplication, under which it is an associative commutative algebra. We shall construct the Schouten concomitant from a natural one-cochain $\tilde{\theta}$ on $\mathscr{A}_0$ by direct analogy with the construction of the Poisson bracket from θ.

Following Hermann (1973) we set up a cochain complex on $\mathscr{A}_0$ as follows. Let $\mathscr{D}(\mathscr{A}_0)$ be the $\mathscr{A}_0$ module of derivations from $\mathscr{A}_0$ to $\mathscr{A}_0$. An r-cochain f is an antisymmetric $\mathscr{A}_0$-multilinear map of $\mathscr{D}(\mathscr{A}_0)X \cdots X\mathscr{D}(\mathscr{A}_0)$ (r times) into $\mathscr{A}_0$. Let $F_r(M)$ be the $\mathscr{A}_0$ module of r-cochains. Define, for $S \in \mathscr{A}_0$, $D \in \mathscr{D}(\mathscr{A}_0)$, the one-cochain $\tilde{d}S$ by $\langle \tilde{d}S, D\rangle = DS$.

For $f \in F_r(M)$, define $\tilde{d}f \in F_{r+1}(M)$ by

$$\tilde{d}f(D_1 \cdots D_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1} D_i f(D_1 \cdots \hat{D}_i \cdots D_{r+1}) + \sum_{1 \leqslant i < j \leqslant r+1} (-1)^{i+j} f([D_i, D_j], D_1 \cdots \hat{D}_i \cdots \hat{D}_j \cdots D_{r+1})$$

Here $\hat{D}_i$ means that D_i is omitted. It follows that $\tilde{d}\tilde{d} = 0$. We shall first obtain $\tilde{\theta}$ by working in one coordinate patch of M with coordinates $q^1, \ldots, q^n$, and then give a coordinate-free definition. Denote the local vector fields on M by $e_i = \partial/\partial q^i$. Then

$$C(e_i) = p_i$$

Let D be a derivation on $\mathscr{A}_0$. Derivations on $\mathscr{A}_0$ are completely defined by their values on the elements q^i and e_i. Define the associated vector field X_D on T^*M by

$$X_D(C(S)) = C(DS)$$

and define the one-cochain $\tilde{\theta}$ on $\mathscr{A}_0$ by

$$C(\langle \tilde{\theta}, D\rangle) = \langle \theta, X_D\rangle \tag{2.2}$$

Now $\theta = p_i\, dq^i$, so

$$\begin{aligned}\langle \theta, X_D\rangle &= p_i\langle dq^i, X_D\rangle = p_i X_D(q^i)\\ &= C(e_i)C(Dq^i) = C(e_i \cap Dq^i)\\ &= C(\langle e_i\, \tilde{d}q^i, D\rangle)\end{aligned}$$

Hence by (2.2),

$$\tilde{\theta} = e_i \, \tilde{d}q^i \tag{2.3}$$

It may help the understanding of the action of $\tilde{\theta}$ if we write

$$Dq^i = S_D^{(i)} \tag{2.4}$$

where $S_D^{(i)} \in \mathscr{A}_0$ has the form

$$S_D^{(i)} = \bigoplus_{m=\infty}^{\infty} S_D^{(i)i_1i_2\cdots i_m}(q)e_{i_1} \cap \cdots \cap e_{i_m} \tag{2.5}$$

Then

$$\begin{aligned}\langle\tilde{\theta}, D\rangle &= e_i \cap Dq^i \\ &= \bigoplus_{m=\infty}^{\infty} S_D^{(i)i_1i_2\cdots i_m}(q)e_i \cap e_{i_1} \cap \cdots \cap e_{i_m}\end{aligned}$$

For example, any vector field $X \in T^{(1)}M$ provides a derivation on $\mathscr{A}_0$, namely, the Lie derivative D_X. We have

$$D_X q^i = X^i, \qquad \langle\tilde{\theta}, D_X\rangle = e_i \cap X^i = X$$

The analogous analysis in T^*M is that X lifts to a vector field $\bar{X}$ on T^*M, and $\langle\theta, \bar{X}\rangle = p_i X^i = C(X)$.

We may describe the action of $\tilde{\theta}$ in a coordinate-free way as follows. Let $\varphi \in T^{(0)}M$; then from (2.4) and (2.5)

$$D\varphi = \varphi_{,i} Dq^i = \left(\bigoplus_m S_D^{(i)i_1\cdots i_m}(q)e_{i_1} \cap \cdots \cap e_{i_m}\right)(e_i\varphi) = S_D\varphi$$

so that D determines the element

$$S_D = \left(\bigoplus_m S_D^{(i)i_1\cdots i_m} e_{i_1} \cap \cdots \cap e_{i_m}\right) \otimes e_i$$

of $\mathscr{A}_0 \otimes T^{(1)}M$. Then we obtain $\langle\tilde{\theta}, D\rangle$ by symmetrizing S_D so as to lie in $\mathscr{A}_0 \cap T^{(1)}M$, and so in $\mathscr{A}_0$.

From equation (3) we have

$$\tilde{d}\tilde{\theta} = \tilde{d}e_i \wedge \tilde{d}q^i$$

$$\tilde{d}\tilde{\theta}(D_1, D_2) = (D_1 e_i) \cap (D_2 q^i) - (D_2 e_i) \cap (D_1 q^i)$$

It is easy to verify that $\tilde{d}\tilde{\theta}$ is nonsingular. For if for all derivations D_2,

$$\tilde{d}\tilde{\theta}(D_1, D_2) = 0$$

then taking $D_2 q^i = \delta_k{}^i$, $D_2 e_i = 0$, we obtain

$$D_1(e_k) = 0$$

and taking $D_2 q^i = 0$, $D_2 e_i = \delta_i{}^k$, we obtain

$$D_1(q^k) = 0$$

Hence D_1 vanishes and $\tilde{d}\tilde{\theta}$ is nonsingular. To each $S \in \mathscr{A}_0$ we may now define a Hamiltonian derivation D_S by the requirement that for all $D \in \mathscr{D}(\mathscr{A}_0)$,

$$\begin{aligned} DS &= \tilde{d}\tilde{\theta}(D_S, D) \\ &= (D_S e_i) \cap (Dq^i) - (De_i) \cap (D_S q^i) \end{aligned}$$

Taking first $Dq^i = \delta_k{}^i$, $De_i = 0$ we obtain

$$D_S e_k = DS = S^{i_1 \cdots i_s}{}_{,k} e_{i_1} \cap \cdots \cap e_{i_s}$$

and then taking $Dq^i = 0$, $De_i = \delta_i{}^k$, we obtain

$$-D_S q^k = DS = sS^{i_1 \cdots i_{s-1} k} e_{i_1} \cap \cdots \cap e_{i_{s-1}}$$

Hence

$$\langle \tilde{\theta}, D_S \rangle = sS$$

and we obtain for the Schouten concomitant

$$\begin{aligned} [S, T] &= \tilde{d}\tilde{\theta}(D_S, D_T) \\ &= -S^{i_1 \cdots i_s}{}_{,k} e_{i_1} \cap \cdots \cap e_{i_s} \cap tT^{j_1 \cdots j_{t-1} k} e_{j_1} \cap \cdots \cap e_{j_{t-1}} \\ &\quad + T^{i_1 \cdots i_t}{}_{,k} e_{i_1} \cap \cdots \cap e_{i_t} \cap sS^{j_1 \cdots j_{s-1} k} e_{j_1} \cap \cdots \cap e_{j_{s-1}} \end{aligned}$$

which agrees with equation (2.1).

We conclude by posing a problem. In a thorough-going formulation of quantum mechanics based only on the algebra of observables, the underlying concept of a C^∞ configuration space would be superfluous. One might develop a quantum mechanics based on any suitable associative commutative algebra of "position operators." This consideration, together with the foregoing analysis, suggests that the following problem merits investigation. Let A be an associative commutative algebra and let $D(A)$ be its Lie algebra of derivations. The symmetric algebra $\mathscr{S}_0 = S(A \oplus \mathscr{D}(A))$ is an associative commutative algebra rather like $\mathscr{A}_0$, so we denote the product operation by $\cap$. Now we can impose a Lie product structure on $\mathscr{S}_0$ which is like the Schouten concomitant on $\mathscr{A}_0$ by taking the semidirect product rules of $A \circledS \mathscr{D}(A)$, namely,

$$[a_1, a_2] = 0, \qquad [D_1, a_1] = D_1 a_1, \qquad [D_1, D_2] = D_1 D_2 - D_2 D_1,$$
$$a_i \in A, \qquad D_i \in \mathscr{D}(A)$$

and extending them to $\mathscr{S}_0$ by distributivity,

$$[S_1, S_2 \cap S_3] = [S_1, S_2] \cap S_3 + S_2 \cap [S_1, S_3], \qquad S_i \in \mathscr{S}_0$$

The problem is: when can we construct this Lie product from a natural one-cochain on $\mathscr{S}_0$?

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